Stochastic User Equilibrium with Equilibrated Choice Sets: Solving the Restricted SUE with Threshold and large-scale application

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Outline

• Motivation
• Framework
• Algorithm
• Large-scale case-study
• Conclusions
Motivation

• Solution algorithms exist for large-scale application of DUE
  – efficient, e.g. by avoiding simulation
  – convergence can be verified from gap measure

• Solution algorithms exist for large-scale application of SUE
  – only approximate SUE
  – explicit/implicit choice set generation (i.e. subset of all possible routes)
  – often apply simulation in path generation and/or flow allocation

• Solution algorithm for large-scale application of RSUET
  – efficient
  – generic
  – avoid simulation
  – convergence should be easy to verify
Solution algorithm framework

• Given Φ and Ω then the route flow $x \in G$ is a RSUET($\Phi, \Omega$) if and only if for all $r \in R_m$ and $m = 1, 2, \ldots, M$:

$$x_{mr} > 0 \Rightarrow r \in \tilde{R}_m \land x_{mr} = d_m \cdot P_{mr}(c(x)|\tilde{R}_m) \land c_{mr}(x) \leq \Omega\{c_{ms}(x) : s \in \tilde{R}_m\}$$

$$x_{mr} = 0 \Rightarrow r \notin \tilde{R}_m \land c_{mr}(x) \geq \Phi\{c_{ms}(x) : s \in \tilde{R}_m\}$$

• Solution algorithm which decomposes path generation and path loading
  – RSUET model formulated with one condition concerning the distribution of flow and one posing a cost restriction on unused paths


**Solution algorithm framework**

- Given $\Phi$ and $\Omega$ then the route flow $\mathbf{x} \in G$ is a RSUET($\Phi, \Omega$) if and only if for all $r \in R_m$ and $m = 1,2,...,M$:

$$
\begin{align*}
x_{mr} > 0 & \Rightarrow r \in \tilde{R}_m \land x_{mr} = d_m \cdot P_m(\mathbf{c}(\mathbf{x})|\tilde{R}_m) \land c_{mr}(\mathbf{x}) \leq \Omega(\{c_{ms}(\mathbf{x}) : s \in \tilde{R}_m\}) \\
x_{mr} = 0 & \Rightarrow r \notin \tilde{R}_m \land c_{mr}(\mathbf{x}) \geq \Phi(\{c_{ms}(\mathbf{x}) : s \in \tilde{R}_m\})
\end{align*}
$$

- **Path-based solution algorithm**
  - RSUET model formulated in space of paths
  - path-based measures can be included in utility function for flow allocation
  - memory issues related to path storage seem not a concern in modern computers
Solution algorithm

- **Step 0: Initialisation**

- **Step 1: Column generation phase.** ‘Systematically’ grow choice set to ensure fulfilment of second condition at convergence

- **Step 2: Restricted master problem phase.** Distribution of flow among used paths according to choice model to ensure fulfilment of first condition at convergence

- **Step 3: Network loading phase.** Perform network loading to get link flows, link costs and path costs resulting from the path flows

- **Step 4: Threshold condition phase.** Check if threshold condition is violated for any OD-pair. Remove violating paths, redistribute flow and perform network loading

- **Step 5: Convergence evaluation phase.** Apply consistent convergence criterion to evaluate whether both equilibrium conditions are fulfilled within a threshold. If not: Return to step 1.
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### Solution algorithm: Column generation phase

**Step 1** *Column generation phase.* Let $k_{m,n-1}$ denote the current number of unique paths in the choice set of used paths for OD-pair $m=1, 2, \ldots, M$ in iteration $n-1$.

<table>
<thead>
<tr>
<th>For RSUET(min,$\Omega$):</th>
<th>For RSUET($\Phi,\Omega$):</th>
<th>For RSUET(max,$\Omega$):</th>
</tr>
</thead>
<tbody>
<tr>
<td>For each origin, perform a shortest path search to all destinations based on actual link travel costs $t_s(f_{n-1})$. If for any OD-pair $m=1, 2, \ldots, M$ a new unique path $i$ is generated, add it to the choice set $\tilde{R}<em>{m,n}$ with flow $x</em>{mi,n-1} = 0$.</td>
<td>For each OD-pair $m \in M$, based on actual link travel costs $t_s(f_{n-1})$, check for a new route to add to the choice set $\tilde{R}<em>{m,n}$ by applying some path generation method which supports the fulfillment of the $\Phi$ operator. If for any OD-pair $m=1, 2, \ldots, M$ a new unique path $i$ is generated, add it to the choice set $\tilde{R}</em>{m,n}$ with flow $x_{mi,n-1} = 0$; if several routes possible, add only the shortest one.</td>
<td>Perform $k_{m,n-1}$-shortest path search for each OD-pair $m=1, 2, \ldots, M$ based on actual link travel costs $t_s(f_{n-1})$. If for any OD-pair $m=1, 2, \ldots, M$ a new unique path $i$ is generated among the $k$ generated paths, add it to the choice set $\tilde{R}<em>{m,n}$ with flow $x</em>{mi,n-1} = 0$; if several new unique paths are generated, add only the shortest one.</td>
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</table>

- Relates to the second RSUET condition
  \[
  x_{mr} = 0 \Rightarrow r \notin \tilde{R}_m \land c_{mr}(x) \geq \Phi( \{c_{ms}(x) : s \in \tilde{R}_m \})
  \]

- Choice set is ‘systematically’ grown
Solution algorithm

• **Step 0: Initialisation**

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Solution algorithm: Restricted Master Problem phase

| Step 2 | Restricted master problem phase. Given the choice sets \( \tilde{R}_{m,n} \) for all \( m=1, 2, \ldots, M \), apply the selected inner assignment component and averaging scheme to find the new flow solution \( x_n \).

- Relates to the first RSUET condition

\[
x_{mr} > 0 \quad \Rightarrow \quad r \in \tilde{R}_m \quad \land \quad x_{mr} = d_m \cdot P_{mr}(c(x) | \tilde{R}_m) \quad \land \quad c_{mr}(x) \leq \Omega\left(\{c_{ms}(x): s \in \tilde{R}_m\}\right)
\]

- Inner assignment component
  - ordinary path-based SUE flow allocation methods
  - efficient path-based DUE methods using cost transformation function for closed-form Logit type choice models
    \[
    \tilde{c}_{mr}(x) = x_{mr} \cdot \exp(\theta \cdot c_{mr}(x))
    \]
Solution algorithm: Restricted Master Problem phase

| Step 2 | Restricted master problem phase. Given the choice sets \( \tilde{R}_{m,n} \) for all \( m=1, 2, \ldots, M \), apply the selected inner assignment component and averaging scheme to find the new flow solution \( x_n \).

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- Inner assignment component

  - ordinary path-based SUE flow allocation methods
  - efficient path-based DUE methods using cost transformation function for closed-form Logit type choice models

\[
\tilde{c}_{mr}(x) = x_{mr} \cdot \exp(\theta \cdot c_{mr}(x))
\]

\[
P_{mr}(c(x)|\tilde{R}_m) = \frac{x_{mr}}{\sum_{s \in \tilde{R}_m} x_{ms}} = \frac{\tilde{c}_{mr}(x)}{\sum_{s \in \tilde{R}_m} \frac{\tilde{c}_{ms}(x)}{\exp(\theta \cdot c_{ms}(x))}}
\]

\[
\tilde{c}_{mr}(x) = \tilde{c}_{ms}(x), \forall r, s \in \tilde{R}_m
\]

\[
P_{mr}(c(x)|\tilde{R}_m) = \frac{x_{mr}}{\sum_{s \in \tilde{R}_m} x_{ms}} = \frac{\tilde{c}_{mr}(x)}{\sum_{s \in \tilde{R}_m} \frac{\tilde{c}_{ms}(x)}{\exp(\theta \cdot c_{ms}(x))}}
\]

\[
= \frac{\tilde{c}_{mr}(x) \cdot \exp(\theta \cdot c_{mr}(x))}{\sum_{s \in \tilde{R}_m} \exp(\theta \cdot c_{ms}(x))} = \exp(-\theta \cdot c_{ms}(x))
\]

\[
= \frac{\tilde{c}_{mr}(x) \cdot \sum_{s \in \tilde{R}_m} \frac{1}{\exp(\theta \cdot c_{ms}(x))}}{\sum_{s \in \tilde{R}_m} \exp(-\theta \cdot c_{ms}(x))}
\]
Solution algorithm: Restricted Master Problem phase

| Step 2 | Restricted master problem phase. Given the choice sets $\tilde{R}_{m,n}$ for all $m=1, 2, \ldots, M$, apply the selected inner assignment component and averaging scheme to find the new flow solution $\mathbf{x}_n$. |

• Relates to the first RSUET condition

$$x_{mr} > 0 \Rightarrow r \in \tilde{R}_m \land x_{mr} = d_m \cdot P_{mr}(c(\mathbf{x})|\tilde{R}_m) \land c_{mr}(\mathbf{x}) \leq \Omega(\{c_{ms}(\mathbf{x}) : s \in \tilde{R}_m\})$$

• Inner assignment component
  – ordinary path-based SUE flow allocation methods
  – efficient path-based DUE methods using cost transformation function for closed-form Logit type choice models
    $$\tilde{c}_{mr}(\mathbf{x}) = x_{mr} \cdot \exp(\theta \cdot c_{mr}(\mathbf{x}))$$

• Averaging scheme
  – MSA, MSWA, Self-regulated averaging, Armijo, ...
Solution algorithm

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Solution algorithm: Threshold condition phase

<table>
<thead>
<tr>
<th>Step 4</th>
<th>Set $m=1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 4.1</td>
<td>For each route $r$ in the choice set $\tilde{R}<em>{m,n}$, check whether the threshold condition $c</em>{mr}(x_n) \leq \Omega({c_{ms}(x): s \in \tilde{R}<em>{m,n} }; s_m)$ is violated. If any route $r$ violates this condition, and if $n \geq K</em>{min}$, then flag the route that violates the threshold condition the most.</td>
</tr>
<tr>
<td>Step 4.2</td>
<td>If no route is flagged by Step 4.1 and if $m &lt; M$, set $m = m+1$ and return to Step 4.1. If no routes are flagged by Step 4.1 and if $m = M$, continue to Step 4.3. If a route $r$ is flagged by Step 4.1, remove the route from the choice set and redistribute flow $x_{mr,n}$ among the remaining currently-used routes $s$ according to the following: $x_{ms,n} = x_{ms,n} + x_{mr,n} \cdot \frac{x_{mr,n}}{d_m - x_{mr,n}}$. If $m &lt; M$, set $m = m+1$ and return to Step 4.1. If $m = M$, continue.</td>
</tr>
<tr>
<td>Step 4.3</td>
<td>If no routes have been removed for any of the $M$ OD-pairs, continue. Else, perform the network loading, compute the link travel costs $t_a(f_a)$, the generalised path travel costs $C(X_n)$ and (if relevant/included) the path-size factors.</td>
</tr>
</tbody>
</table>

- Relates to the first RSUET condition

\[ x_{mr} > 0 \quad \Rightarrow \quad r \in \tilde{R}_m \quad \wedge \quad x_{mr} = d_m \cdot P_{mr}(c(x) | \tilde{R}_m) \quad \wedge \quad c_{mr}(x) \leq \Omega(\{c_{ms}(x): s \in \tilde{R}_m \}) \]

- Lower risk of ‘destabilisation’: drop max one route per iteration per OD-pair and only after $K_{min}$ iterations
Solution algorithm

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Solution algorithm: Convergence evaluation phase

| Step 5 | Convergence evaluation phase. If the gap measure consisting of the sum of Rel.Gap$_n^{used}$ and Rel.Gap$_n^{unused}$ is below a pre-specified threshold $\xi$, Stop. Else, set $n=n+1$ and return to Step 1. |

- Utilise cost transformation function to consistently evaluate convergence
  - first condition (distribution of flow)

$$Rel.Gap_{n}^{used} = \frac{\sum_{m=1}^{M} \sum_{r \in R_m} x_{mr,n} \cdot (\tilde{c}_{mr}(\mathbf{x}_n) - \tilde{c}_{m,min}(\mathbf{x}_n))}{\sum_{m=1}^{M} \sum_{r \in R_m} x_{mr,n} \cdot \tilde{c}_{mr}(\mathbf{x}_n)}$$
Solution algorithm: Convergence evaluation phase

Step 5: Convergence evaluation phase. If the gap measure consisting of the sum of

\[ \text{Rel.Gap}_n^{\text{Used}} \text{ and } \text{Rel.Gap}_n^{\text{Unused}} \]

is below a pre-specified threshold \( \xi \), Stop. Else, set \( n = n+1 \) and return to Step 1.

- Utilise cost transformation function to consistently evaluate convergence

  - first condition (distribution of flow)

  \[ \text{Rel.Gap}_n^{\text{Used}} = \frac{\sum_{m=1}^{M} \sum_{r \in R_m} x_{mr,n} \cdot (\tilde{c}_{mr}(x_n) - \hat{c}_{m,\text{min}}(x_n))}{\sum_{m=1}^{M} \sum_{r \in R_m} x_{mr,n} \cdot \tilde{c}_{mr}(x_n)} \]

  - second condition (composition of choice set)

  - RSUET(min, \( \Omega \)):

  \[ \text{Rel.Gap}_n^{\text{Unused}} = \frac{\sum_{m=1}^{M} d_m \cdot (\min_{r \in R_m, x_{mr} > 0} (c_{mr}(x_n)) - \min_{r \in R_m, x_{mr} > 0} (c_{mr}(x_n)))}{\sum_{m=1}^{M} d_m \cdot \min_{r \in R_m, x_{mr} > 0} (c_{mr}(x_n))} \]

  - RSUET(max, \( \Omega \)):

  \[ \text{Rel.Gap}_n^{\text{Unused}} = \frac{\sum_{m=1}^{M} d_m \cdot (\max_{r \in R_m, x_{mr} > 0} (c_{mr}(x_n)) - c_{mr,k}(x_n))}{\sum_{m=1}^{M} d_m \cdot \max_{r \in R_m, x_{mr} > 0} (c_{mr}(x_n))} \]
Zealand application of RSUET(min, $\tau\cdot$min)

• Network and demand
  – 3.2 million trips, 19 user classes, 3 vehicle types
  – 18,706 one-directional links, 429 zones

• Restricted master problem phase
  – cost function considering free-flow travel time, congested travel time and length
  – MultiNomial Logit (MNL) and Path-Size Logit (PSL)
  – inner assignment component
    • DUE Path Swap
    • Inner Logit
  – averaging scheme
    • Method of Successive Weighted Averages (MSWA)
      $$\gamma_n = \frac{n^d}{1^d + 2^d + \ldots + n^d}$$
Zealand application of RSUET(min, τ·min)

- Threshold condition phase

\[ \Omega\left(\{c_{ms}(x) : s \in \bar{R}_m \}\right) = 1.2 \cdot \min\{c_{ms}(x) : s \in \bar{R}_m \} \]

- 16,618 observed paths from GPS traces
- 1,169 observed link flows on network
Results

• Fast convergence for MNL and PSL RSUET(min,1.2·min)

• Step-size parameter $d$ influences convergence speed

• Robust towards congestion level
  – slightly slower convergence
  – larger choice sets
  – increased effect of threshold condition
Results

• Fast convergence and very reasonable choice set size
  – final choice sets generated within first few iterations
  – flow allocation converges very fast

• Comparison to GPS data: observed paths are generally reproduced

• Comparison to link flows: very successful reproduction

<table>
<thead>
<tr>
<th>θ</th>
<th>Choice set size</th>
<th>Coverage, λ=0.8</th>
<th>Efficiency index</th>
<th>Excluded paths</th>
<th>R² link flow</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Min.</td>
<td>Avg.</td>
<td>Max.</td>
<td>Ite 25</td>
<td>Ite 100</td>
</tr>
<tr>
<td>0.05</td>
<td>1</td>
<td>2.367</td>
<td>10</td>
<td>0.8431</td>
<td>0.8431</td>
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<tr>
<td>0.1</td>
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<td>2.484</td>
<td>10</td>
<td>0.8452</td>
<td>0.8452</td>
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<tr>
<td>0.2</td>
<td>1</td>
<td>2.695</td>
<td>12</td>
<td>0.8487</td>
<td>0.8487</td>
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<td>1.0</td>
<td>1</td>
<td>3.057</td>
<td>13</td>
<td>0.8548</td>
<td>0.8548</td>
</tr>
</tbody>
</table>
Results

- MultiNomial Probit (MNP) from Danish National Transport Model
- Increased coverage at cost of slow convergence
  - choice sets continue to grow
  - not stable link flows

<table>
<thead>
<tr>
<th>Choice set size</th>
<th>Coverage, ( \lambda = 0.8 )</th>
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<th>Excluded paths</th>
<th>( R^2 ) link flow</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>Avg.</td>
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<tr>
<td>( \theta = 0.05 )</td>
<td>1</td>
<td>2.367</td>
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<td>( \theta = 0.1 )</td>
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<td>1</td>
<td>3.057</td>
<td>13</td>
<td>0.8548</td>
</tr>
<tr>
<td>MNP SUE</td>
<td>1</td>
<td>14.894</td>
<td>100</td>
<td>0.8959</td>
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<tr>
<td>Mixed MNP SUE</td>
<td>1</td>
<td>25.365</td>
<td>100</td>
<td>0.8959</td>
</tr>
</tbody>
</table>
Conclusions

• Generic path-based RSUET($\Phi,\Omega$) solution algorithm
  – allows use of existing SUE flow allocation methods among used paths
  – allows use of efficient DUE methods for certain Logit-type choice
    models (cost transformation function)
  – allows consistent evaluation of convergence

• Application to large-scale case study
  – very fast convergence to equilibrated solution (across scale
    parameters, demand levels and step-size strategies)
  – reproduction of observed routes and link counts
References


Thank you for your attention
Results

- Effect of threshold condition,
MNL RSUET(min, 1.2min)

<table>
<thead>
<tr>
<th>Path</th>
<th>Category ID</th>
<th>$l_{ir}$ [km]</th>
<th>$t_{FreeTT, ir(x)}$ [min]</th>
<th>$t_{CongTT, ir(x)}$ [min]</th>
<th>$c_{ir}(x)$</th>
<th>$c_{ir}(x)/c_{i,min}(x)$</th>
<th>Flow [%]</th>
</tr>
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<tbody>
<tr>
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<td>16.44</td>
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</tr>
</tbody>
</table>
Results

- Increase due to recalculation of correction terms in PSL
- Increase with increasing demand/congestion level
  - 90/105/130/145/180 seconds per iteration for scale-parameter 1.0/1.25/1.5/1.75/2.0